3.7C. COROLLARY.
Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Here $a_n = \frac{1}{n^2}$

$$\therefore \sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \left(\frac{1}{(2^n)^2}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \text{ converges}$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is converges

(e) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is converges.

3.7D Theorem: If $\{a_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of positive numbers and if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} n a_n = 0$.

Proof: given $\{a_n\}_{n=1}^{\infty}$ is a non increasing sequence

$a_1 \geq a_2 \geq a_3 \geq \cdots\cdots a_n \geq a_{n+1} \geq \cdots$

Also given $\sum_{n=1}^{\infty} a_n$ converges

Let $s_n = a_1 + a_2 + \cdots + a_n$ $\therefore$ seq of partial sum $\{s_n\}_{n=1}^{\infty}$ is convergent.

$$\lim_{n\to\infty} s_n = A = \lim_{n\to\infty} s_{2n}$$

$$\therefore \lim_{n\to\infty} s_{2n} - s_n = A - A$$

$$s_{2n} - s_n \underset{\lim}{=} (a_1 + a_2 + \cdots + a_n + a_{n+1} + a_{n+2} + \cdots + a_{2n}) - (a_1 + a_2 + \cdots + a_n)$$

$$= 0$$

$$s_{2n} - s_n = a_{n+1} + a_{n+2} + \cdots + a_{2n} \geq a_{2n} + a_{2n} + \cdots + a_{2n}$$

$$s_{2n} - s_n \geq n\, a_{2n}$$

$$\lim_{n\to\infty} (s_{2n} - s_n) \geq \lim_{n\to\infty} n\, a_{2n}$$

$$0 \geq \lim_{n\to\infty} n\, a_{2n}$$

$\Rightarrow \lim_{n\to\infty} n\, a_{2n} = 0$ since $\{a_n\}_{n=1}^{\infty}$ non increasing sequence of positive numbers.

$$\Rightarrow \lim_{n\to\infty} 2n\, a_{2n} = 0 \quad \text{——} \quad (1)$$

But $a_{2n+1} \leq a_{2n}$

$$(2n+1)\, a_{2n+1} \leq \frac{(2n+1)}{2n}\, 2n a_{2n}$$

$$\lim_{n\to\infty} (2n+1)\, a_{2n+1} \leq \lim_{n\to\infty} \left(\frac{2n+1}{2n}\right) \lim_{n\to\infty} 2n\, a_{2n}$$

$$\lim_{n\to\infty} (2n+1)\, a_{2n+1} \leq 0$$

$$\Rightarrow \lim_{n\to\infty} (2n+1)\, a_{2n+1} = 0 \quad \text{——} \quad (2)$$

using (1) and (2)

$$\lim_{n\to\infty} n\, a_n = 0.$$

Sec 3.10 The Class l^2.

Definition: The class l^2 is the class of all sequences $s = \{s_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} s_n^2 < \infty$. (i.e.) $\sum_{n=1}^{\infty} s_n^2$ is cgt.

<u>Ex</u> The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is an element of Class l^2

since $\sum_{n=1}^{\infty} s_n^2 = \sum \frac{1}{n^2} < \infty$

$\therefore$ The seq $\{\frac{1}{n}\}_{n=1}^{\infty}$ is an element of class l^2.

But $\{\frac{1}{\sqrt{n}}\}_{n=1}^{\infty}$ is not an element of l^2

since $s_n = \frac{1}{\sqrt{n}}$ $\quad s_n^2 = \frac{1}{n}$ $\quad \sum_{n=1}^{\infty} s_n^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent.

$\therefore$ The seq $\{\frac{1}{\sqrt{n}}\}$ is not an element of l^2.

Theorem The SCHWARZ INEQUALITY.

statement: If $s = \{s_n\}_{n=1}^{\infty}$ and $t = \{t_n\}_{n=1}^{\infty}$ are in l^2, then $\sum_{n=1}^{\infty} s_n t_n$ is absolutely convergent and

$$\left| \sum_{n=1}^{\infty} s_n t_n \right| \leq \left(\sum_{n=1}^{\infty} s_n^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} t_n^2 \right)^{\frac{1}{2}}.$$

<u>Proof</u> We may assume that there is at least on s_n say s_N not equal to 0. Otherwise the theorem is trivial. For fixed $n \geq N$ and any $x \in R$ we have

$$\sum_{k=1}^{n} (x s_k + t_k)^2 \geq 0.$$

$$x^2\sum_{k=1}^{n} s_k^2 + 2x\sum_{k=1}^{n} s_k t_k + \sum_{k=1}^{n} t_k^2 \geq 0.$$

This can be written $Ax^2+Bx+C \geq 0$, where

$$A = \sum_{k=1}^{n} s_k^2 > 0 \quad B = 2\sum_{k=1}^{n} s_k t_k \quad C = \sum_{k=1}^{n} t_k^2$$

[From calculus we know that the minimum value of Ax^2+Bx+C $(A>0)$ occurs when $x = -\frac{B}{2A}$.

$\therefore$ If we set $x = -\frac{B}{2A}$, we have

$$A\left(\frac{-B}{2A}\right)^2 + B\left(\frac{-B}{2A}\right) + C \geq 0$$

(or)

$$B^2 \leq 4AC$$

(e) $$\left(2\sum_{k=1}^{n} s_k t_k\right)^2 \leq 4 \sum_{k=1}^{n} s_k^2 \sum_{k=1}^{n} t_k^2$$

$$\left(\sum_{k=1}^{n} s_k t_k\right)^2 \leq \left(\sum_{k=1}^{n} s_k^2\right)\left(\sum_{k=1}^{n} t_k^2\right)$$

Taking square root on bothside

$$\sum_{k=1}^{n} s_k t_k \leq \left(\sum_{k=1}^{n} s_k^2\right)^{1/2} \left(\sum_{k=1}^{n} t_k^2\right)^{1/2}$$

$$\sum_{k=1}^{n} |s_k t_k| \leq \left(\sum_{k=1}^{n} s_k^2\right)^{1/2} \left(\sum_{k=1}^{n} t_k^2\right)^{1/2}$$

$$\sum_{k=1}^{\infty} |s_k t_k| \leq \left(\sum_{k=1}^{\infty} s_k^2\right)^{1/2} \left(\sum_{k=1}^{\infty} t_k^2\right)^{1/2} \quad \text{--- } (1)$$

Given $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are in l^2

$\Rightarrow \sum_{n=1}^{\infty} s_n^2$ ~~convergent~~ and $\sum_{n=1}^{\infty} t_n^2$ convergent

$\therefore$ Right hand side of inequality (1) is convergent

$\therefore$ By comparison test

$\sum_{n=1}^{\infty} |s_n t_n|$ is convergent.

$\Rightarrow \sum_{n=1}^{\infty} s_n t_n$ converges absolutely.

and $\sum_{n=1}^{\infty} |s_n t_n| \le \left(\sum_{n=1}^{\infty} s_n^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} t_n^2\right)^{\frac{1}{2}}$.

3.10C Theorem: THE MINKOWSKI INEQUALITY.

statement: If $s = \{s_n\}_{n=1}^{\infty}$ and $t = \{t_n\}_{n=1}^{\infty}$ are in l^2, then $s+t = \{s_n + t_n\}_{n=1}^{\infty}$ is in l^2 and

$$\left[\sum_{n=1}^{\infty} (s_n + t_n)^2\right]^{\frac{1}{2}} \le \left(\sum_{n=1}^{\infty} s_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} t_n^2\right)^{\frac{1}{2}}$$

Proof: By hypothesis, the series $\sum_{n=1}^{\infty} s_n^2$ and $\sum_{n=1}^{\infty} t_n^2$ converges

Also the series $\sum_{n=1}^{\infty} s_n t_n$ converges, by the previous theorem

Since $(s_n + t_n)^2 = s_n^2 + 2 s_n t_n + t_n^2$

$\Rightarrow \sum_{n=1}^{\infty} (s_n + t_n)^2$ Converges and

$$\sum_{n=1}^{\infty} (s_n + t_n)^2 = \sum_{n=1}^{\infty} s_n^2 + 2 \sum_{n=1}^{\infty} s_n t_n + \sum_{n=1}^{\infty} t_n^2$$

Applying the Schwarz inequality to the second sum on the right, we obtain

$$\sum_{n=1}^{\infty}(s_n+t_n)^2 \le \sum_{n=1}^{\infty} s_n^2 + 2\left(\sum_{n=1}^{\infty} s_n^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} t_n^2\right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} t_n^2$$

$$\sum_{n=1}^{\infty}(s_n+t_n)^2 \le \left[\left(\sum_{n=1}^{\infty} s_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} t_n^2\right)^{\frac{1}{2}}\right]^2$$

Taking square root on both side.

$$\left[\sum_{n=1}^{\infty}(s_n+t_n)^2\right]^{\frac{1}{2}} \le \left(\sum_{n=1}^{\infty} s_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} t_n^2\right)^{\frac{1}{2}}$$

Hence proved.

Chapter 4 Limits and metric spaces.